Continuous time-reversal

February 28, 2023

Let $(\mathbf{B}_t)_{t\in[0,T]}$ a d-dimensional Brownian motion associated with the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. We define the process $(\mathbf{X}_t)_{t\in[0,T]}$ such that for any $t\in[0,T]$

$$\mathbf{X}_t = \mathbf{X}_0 + \int_0^t b(s,\mathbf{X}_s)\mathrm{d}s + \mathbf{B}_t.$$

In short, we write $\mathrm{d}\mathbf{X}_t = b(t,\mathbf{X}_t)\mathrm{d}t + \mathrm{d}\mathbf{B}_t$. We assume that $b \in \mathrm{C}^\infty(\mathbb{R}\times\mathbb{R}^d,\mathbb{R}^d)$ and is bounded. The goal of this exercise is to show the *time-reversal* formula in continuous-time. More precisely, define $(\mathbf{Y}_t)_{t\in[0,T]} = (\mathbf{X}_{T-t})_{t\in[0,T]}$. We are going to show that

$$\mathrm{d}\mathbf{Y}_t = \{-b(T-t,\mathbf{Y}_t) + \nabla \log p_{T-t}(\mathbf{Y}_t)\}\mathrm{d}t + \mathrm{d}\mathbf{B}_t. \tag{1}$$

In particular, we are going to show that for any $f\in\mathrm{C}_c^\infty(\mathbb{R}^d,\mathbb{R})$, $(\mathbf{M}_t^f)_{t\in[0,T]}$ is a $(\mathbf{Y}_t)_{t\in[0,T]}$-martingale where for any $t\in[0,T]$

$$\mathbf{M}_t^f = f(\mathbf{Y}_t) - f(\mathbf{Y}_0) - \int_0^t \{\langle -b(T-s,\mathbf{Y}_s) + \nabla\log p_{T-s}(\mathbf{Y}_s), \nabla f(\mathbf{Y}_s)\rangle + \tfrac{1}{2}\Delta f(\mathbf{Y}_s)\}\mathrm{d}s,$$

where p_t is the density of the distribution of $\mathbf{X}_t$ (w.r.t. the Lebesgue measure) which is assumed to exist. We recall that $(\mathbf{M}_t^f)_{t\in[0,T]}$ is a $(\mathbf{Y}_t)_{t\in[0,T]}$-martingale if

1. Finite expectation: for any $t\in[0,T]$, $\mathbb{E}[\|\mathbf{M}_t^f\|] < +\infty$,
2. Conditional expectation[1]: for any $s,t\in[0,T]$ with $s\le t$, $\mathbb{E}[\mathbf{M}_t^f|\mathbf{Y}_s] = \mathbf{M}_s^f$

The fact that $(\mathbf{M}_t^f)_{t\in[0,T]}$ is a $(\mathbf{Y}_t)_{t\in[0,T]}$-martingale is equivalent to the fact that $(\mathbf{Y}_t)_{t\in[0,T]}$ is a *weak solution* to (1).

Remark: in what follows we assume that $(t,x)\mapsto\|\nabla\log p_t(x)\|$ has at most linear growth[2]and $(t,x)\mapsto p_t(x)\in\mathrm{C}^\infty([0,T]\times\mathbb{R}^d,\mathbb{R})$. In addition, we assume that $s,t,x_s,x_t\mapsto p_{t|s}(x_t|x_s)\in\mathrm{C}^\infty(\mathsf{A}\times\mathbb{R}^d\times\mathbb{R}^d)$ and is bounded, where $\mathsf{A}=\{(s,t)\,:\, s,t\in[0,T],\ t\ge s\}$. In addition, we have assume that for any $s,t\in\mathsf{A}$ and $x_t\in\mathbb{R}^d$, $x_s\mapsto\|\nabla_{x_s}\log p_{t|s}(x_t|x_s)\|$ has at most linear growth.

We recall the Itô formula. For any $\varphi\in\mathrm{C}^\infty([0,T]\times\mathbb{R}^d,\mathbb{R})$ such that $\|\nabla\log\varphi\|$ has linear growth we have for any $t,s\in[0,T]$

$$\mathbb{E}[\varphi(t,\mathbf{X}_t)-\varphi(s,\mathbf{X}_s)|\mathbf{X}_s] = \mathbb{E}[\int_s^t\{\partial_u\varphi(u,\mathbf{X}_u) + \langle b(u,\mathbf{X}_u),\nabla\varphi(u,\mathbf{X}_u)\rangle + \tfrac{1}{2}\Delta\varphi(u,\mathbf{X}_u)\}\mathrm{d}u|\mathbf{X}_s].$$

[1] Here I have assumed without proof that $(\mathbf{Y}_t)_{t\in[0,T]}$ is Markov

[2] A function $f:\ [0,T]\times\mathbb{R}^d\to\mathbb{R}$ is said to have linear growth if there exists $C\ge 0$ such that for any $t\in[0,T]$ and $x\in\mathbb{R}^d$, $\|f(t,x)\|\le C(1+\|x\|)$.

We also recall the following result. For any $F \in \mathrm{C}_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and $g \in \mathrm{C}^\infty(\mathbb{R}^d, \mathbb{R})$

$$\int_{\mathbb{R}^d} \langle F(x), \nabla g(x) \rangle \mathrm{d}x = - \int_{\mathbb{R}^d} g(x) \mathrm{div}(g)(x) \mathrm{d}x.$$

We denote by $\mathrm{C}_c^\infty(\mathbb{R}^d, \mathbb{R})$, the set of infinitely differentiable continuous functions on $\mathbb{R}^d$ with compact support.

Question 1: Prove that $t \in [0, T]$, $\mathbb{E}[\|\mathbf{M}_t^f\|] < +\infty$.

Question 2: Prove that $(\mathbf{M}_t^f)_{t \in [0,T]}$ is a $(\mathbf{Y}_t)_{t \in [0,T]}$-martingale if and only for any $g \in \mathrm{C}^\infty(\mathbb{R}^d, \mathbb{R})$ and $t, s \in [0, T]$ with $t \geq s$

$$\mathbb{E}[(\mathbf{M}_t^f - \mathbf{M}_s^f) g(\mathbf{Y}_s)] = 0$$

Question 3: Prove that $(\mathbf{M}_t^f)_{t \in [0,T]}$ is a $(\mathbf{Y}_t)_{t \in [0,T]}$-martingale if and only for any $g \in \mathrm{C}^\infty(\mathbb{R}^d, \mathbb{R})$ and $t, s \in [0, T]$ with $t \geq s$

$$\mathbb{E}[g(\mathbf{X}_t) f(\mathbf{X}_t) - g(\mathbf{X}_t) f(\mathbf{X}_s)] = \mathbb{E}[g(\mathbf{X}_t) \int_s^t \{\langle b(u, \mathbf{X}_u) - \nabla \log p_u(\mathbf{X}_u), \nabla f(\mathbf{X}_u) \rangle - \tfrac{1}{2} \Delta f(\mathbf{X}_u)\} \mathrm{d}u].$$

For any $g \in \mathrm{C}_c^\infty(\mathbb{R}^d, \mathbb{R})$ and $t \in [0, T]$, denote $h^{g,t} : \ [0, t] \times \mathbb{R}^d \to \mathbb{R}$ given for any $s \in [0, t]$ and $x \in \mathbb{R}^d$ by

$$h^{g,t}(s, x) = \mathbb{E}[g(\mathbf{X}_t) | \mathbf{X}_s = x].$$

In what follows, we fix $t \in [0, T]$ and $g \in \mathrm{C}_c^\infty(\mathbb{R}^d)$.

Question 4: Show that $h^{g,t} \in \mathrm{C}^\infty([0, t] \times \mathbb{R}^d, \mathbb{R})$.

Question 5: Show that for any $u, s \in [0, t]$ with $u \geq s$ and $\Psi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$

$$\mathbb{E}[\Psi(\mathbf{X}_s)\{h^{g,t}(u, \mathbf{X}_u) - h^{g,t}(s, \mathbf{X}_s) - \int_s^u \{\partial_w h^{g,t}(w, \mathbf{X}_w) + \langle b(w, \mathbf{X}_w), \nabla h^{g,t}(w, \mathbf{X}_w) \rangle + \tfrac{1}{2} \Delta h^{g,t}(w, \mathbf{X}_w)\} \mathrm{d}w\}] = 0$$

Question 6: Show that for any $s \in [0, t]$ and $x \in \mathbb{R}^d$, $\partial_s h^{g,t}(s, x) + \langle b(s, x), \nabla h^{t,g}(s, x) \rangle + \frac{1}{2} \Delta h^{g,t}(s, x) = 0$.

Question 7: Show that

$$\mathbb{E}[g(\mathbf{X}_t) f(\mathbf{X}_t) - g(\mathbf{X}_t) f(\mathbf{X}_s)] = \mathbb{E}[\int_s^t \{f(\mathbf{X}_u) \partial_u h^{g,t}(u, \mathbf{X}_u) + \langle b(u, \mathbf{X}_u), \nabla(h^{g,t}(u, \cdot) f)(\mathbf{X}_u) + \tfrac{1}{2} \Delta(h^{g,t}(u, \cdot) f)\} \mathrm{d}u].$$

Question 8: Show that

$$\mathbb{E}[g(\mathbf{X}_t) f(\mathbf{X}_t) - g(\mathbf{X}_t) f(\mathbf{X}_s)] = \mathbb{E}[\int_s^t \{h^{g,t}(u, \cdot) \langle b(u, \mathbf{X}_u), \nabla f(\mathbf{X}_u) \rangle + h^{g,t}(u, \mathbf{X}_u) \tfrac{1}{2} \Delta f(\mathbf{X}_u) + \langle \nabla f(\mathbf{X}_u), \nabla h^{g,t}(u, \mathbf{X}_u) \rangle \} \mathrm{d}$$

Question 9: Show that

$$\mathbb{E}[\int_s^t \langle \nabla f(\mathbf{X}_u), \nabla h^{g,t}(u, \mathbf{X}_u) \rangle \mathrm{d}u] = -\mathbb{E}[\int_s^t \{\Delta f(\mathbf{X}_u) + \langle \nabla \log p_u(\mathbf{X}_u), \nabla f(\mathbf{X}_u) \rangle h^{g,t}(u, \mathbf{X}_u) \mathrm{d}u].$$

Question 10: Conclude the proof.